

# The Hausdorff Dimension of Surfaces in Two-Dimensional Quantum Gravity Coupled to Ising Minimal Matter

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## Abstract

Within the framework of string field theory the intrinsic Hausdorff dimension  $d_H$  of the ensemble of surfaces in two-dimensional quantum gravity has recently been claimed to be  $2m$  for the class of unitary minimal models ( $p = m + 1, q = m$ ). This contradicts recent results from numerical simulations, which consistently find  $d_H \approx 4$  in the cases  $m = 2, 3, 5$  and  $\infty$ . The string field calculations rely on identifying the scaling behavior of geodesic distance and area with respect to a common length scale  $l$ . This length scale is introduced by formulating the models on a disk with fixed boundary length  $l$ . In this paper we study the relationship between the mean area and the boundary length for pure gravity and the Ising model coupled to gravity. We discuss how this relationship is modified by relevant perturbations in the Ising model. We discuss how this leads to a modified value for the Hausdorff dimension.

## 1 Introduction

An important characterization of surfaces in quantum gravity is the *fractal* or Hausdorff dimension  $d_H$ . The Hausdorff dimension governs the power-law relation between two reparametrization-invariant quantities with dimensions of volume and length;  $V \propto r^{d_H}$ . To introduce a quantity with the dimension of length it is convenient to use the notion of geodesic distance between two points on the surface. In two dimensions the above relation tells us how the area within geodesic distance  $r$  from a marked point scales with  $r$ . It is important to realize that since we integrate over all metrics in quantum gravity, the above notion of area has to be replaced by its average value.

Since a lattice possess a fundamental length scale, it is natural to use the above relation to measure the Hausdorff dimension in numerical simulations. This was

done recently [1, 2, 3], using finite size scaling, for the case of pure two dimensional gravity, and for matter fields (Ising and 3-state Potts model) coupled to gravity. For these models the Hausdorff dimension appears to be universal and have the value 4. It is surprising that there is no apparent evidence of the back-reaction of matter on the Hausdorff dimension; in contrast the string susceptibility exponent is known to depend strongly on the central charge of the matter system. This universality does not seem to be valid for matter systems with central charge  $c > 1$  coupled to gravity — for  $c$  sufficiently large  $d_H$  approaches 2 [2].

The analytical predictions for the Hausdorff dimension, on the other hand, are wide and varied [4, 5, 6, 7]. In the continuum formulation there is no natural notion of a length scale — it has to be introduced by hand. Both the concept of geodesic distance and area are then defined relative to this length scale. All of this may be accomplished by formulating the model on a manifold with a boundary of fixed length, such as the disk. Using either the continuum Liouville or the matrix model formulation one can determine the area scaling behavior of the disk amplitude with boundary length  $l$  [8, 9]. Then either using a transfer matrix formulation [10, 11], or a string field theory formulation [6], one can calculate the scaling behavior of the boundary loop length with respect to an evolution parameter. It can be argued that the evolution parameter defines a notion of geodesic distance on the surface. These two relations in principle determine the Hausdorff dimension. In the case of pure gravity, where a complete transfer matrix formalism exists, one finds Hausdorff dimension 4. Recently a string field theory describing the  $c < 1$   $m$ -th minimal model coupled to gravity has been formulated [6, 12, 13, 14, 15]. This formulation, combined with the assumption of “classical” area scaling  $A \sim l^2$ , leads to a prediction  $d_H = 2m$ . This result has also been obtained in the loop gas formulation of the string field theory Hamiltonian [7].

In this paper we will argue that the scaling of area with boundary length depends crucially on the approach to the continuum limit in the case of matter coupled to gravity. We show in the case of the Ising model how the perturbation by the thermal operator changes the dependence of the mean area on the boundary length, and this in turn changes the Hausdorff dimension from  $d_H = 6$  to  $d_H = 4$ . This phenomena is caused by the subtle interplay between the manner in which the matter and gravitational correlation lengths diverge in the thermodynamic limit.

In Section 2 we derive the scaling of geodesic distance with boundary length, which follows from the string field theory Hamiltonian for  $c < 1$  non-critical strings. In Section 3 we use matrix model techniques to compute the disk amplitude and the resulting scaling of the mean area with boundary length for pure gravity and the Ising model coupled to gravity. In Section 4 we discuss the derivation of the Hausdorff dimension for these two models. Finally Section 5 summarizes our results.

## 2 String field theory

In this section we provide a brief review of the non-critical string theory for  $c < 1$  matter [6, 12, 14]. This will enable us to extract the scaling of geodesic distance with boundary length. For simplicity we will describe the case  $c = 0$ .

An essential feature of both the transfer matrix formulation, and the string field theory, is the choice of a particular ADM or temporal type gauge, in which time corresponds to geodesic distance  $r$ . The Hamiltonian which evolves a loop of length  $l$  (corresponding to a spatial hyper-surface) is assumed to be

$$\mathcal{H}_{disk} = \int_0^\infty dl_1 \int_0^\infty dl_2 \psi^\dagger(l_1) \psi^\dagger(l_2) \psi(l_1 + l_2) (l_1 + l_2) + \int_0^\infty dl \rho(l) \psi(l), \quad (1)$$

where  $\psi^\dagger(l)$  and  $\psi(l)$  are creation and annihilation operators for a loop of length  $l$ , which satisfy canonical commutation relations  $[\psi(l), \psi^\dagger(l')] = \delta(l - l')$ . The first term corresponds to a cubic string vertex and the second term to the annihilation of a single string (a tadpole term). The absence of a kinetic term follows from a few simple assumptions [6]. The disk partition function  $w(l)$  is defined by

$$w(l, \mu) = \lim_{r \rightarrow \infty} \langle 0 | e^{-r \mathcal{H}_{disk}} | 0 \rangle, \quad (2)$$

where  $\mu$  is the cosmological constant. The kernel of the tadpole term  $\rho(l)$  is determined by matching the disk amplitude, computed from the above Hamiltonian, with the result from matrix model calculations. One can show subsequently that the resulting Hamiltonian satisfies the correct Schwinger–Dyson equations.

This formalism has been generalized to include all the  $(p, q)$ -models. The operators  $\psi^\dagger(l)$  and  $\psi(l)$  now correspond to the creation and annihilation of a loop of length  $l$  with a *fixed* uniform matter configuration. The cubic vertex term in the Hamiltonian Eq. (1) must be generalized to include splitting of a loop into two loops with different matter configurations.

From Eq. (1) it follows by dimensional analysis that

$$\mathcal{H}_{disk} \sim l^{[\psi^\dagger(l)]+2}, \quad (3)$$

where  $[\psi^\dagger(l)]$  denotes the boundary length scaling dimension of the loop creation operator. From Eq. (2) this equals the scaling dimension of the disk amplitude,<sup>1</sup> which, for  $(p, q)$  conformal matter, is well known to be [16]:

$$[\psi^\dagger(l)] = [w(l)] = -\left(\frac{p+q}{q}\right). \quad (5)$$

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<sup>1</sup>The relation between  $\psi^\dagger(l)$  and  $w(l)$  is even clearer if one considers the effective Hamiltonian  $\mathcal{H}'$  for processes in which one tracks only a single loop. In [12] it was shown that

$$\mathcal{H}'_{disk} = 2 \int_0^\infty dl_1 \int_0^\infty dl_2 \psi^\dagger(l_1) w(l_2) \psi(l_1 + l_2) (l_1 + l_2). \quad (4)$$

Combining the last two equations we find

$$r \sim l^{(p-q)/q}, \quad (6)$$

which, in the case of the  $m$ -th unitary minimal model yields

$$r \sim l^{1/m}. \quad (7)$$

A different derivation of this result has been given in a loop gas formulation, generalized to include open strings [7].

### 3 Scaling limits of unitary minimal models

In this section we want to relate the mean area  $\bar{A}$  of a disk to its boundary length  $l$ . We will do this by using the matrix model formulation of the disk amplitude. We will show in Section 3.2 that this relation depends on how the continuum limit is taken. Combined with the result of Section 2,  $l \sim r^m$ , this gives the Hausdorff dimension  $d_H$ .

A general  $(p, q)$ -model, with central charge  $c = 1 - 6(p - q)^2/pq$ , coupled to gravity can be defined in terms of two differential operators  $P$  and  $Q$ , of degrees  $p$  and  $q$  respectively, satisfying the string equation  $[P, Q] = 1$  (for a comprehensive review see [17]). To obtain a surface with boundary length  $b$ , on the lattice, we need to compute<sup>2</sup>  $w(b) = \langle \text{tr } \phi^b \rangle$ . In the continuum limit, in the spherical approximation, this becomes

$$w(l, \mu) = \int_{\mu}^{\infty} dx \langle x | e^{+lQ} | x \rangle, \quad (8)$$

where  $\mu$  is a cut-off identified with the cosmological constant and  $b$  has been taken to infinity so as to obtain a finite boundary length  $l$ .

To evaluate the integral in Eq. (8) we need, for a particular model, both the explicit form of  $Q$  and the string equation. For the pure gravity and the Ising model, which we consider in this paper, the operators  $P$  and  $Q$ , together with the string equation, may be found in [17, 18]. We will calculate  $w(l, \mu)$  for these two models in the following subsections. This will give us the desired scaling relations  $\bar{A} \sim l^{\beta}$ .

#### 3.1 Pure gravity ( $m = 2$ )

The continuum limit of the one-matrix model, at its  $k$ -th multi-critical point, describes a  $(2k - 1, 2)$ -matter system coupled to gravity. In particular for  $k = 2$  we obtain pure gravity ( $c = 0$ ). In this case  $Q = d^2 - u(x)$  is the Schrödinger operator and the potential  $u(x)$  is the specific heat. The string equation, in the planar limit,

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<sup>2</sup>Using this definition we obtain a disk with fixed boundary condition for the matter fields. For generalizations to arbitrary boundary conditions see [19]. These formulations, on the other hand, do not allow us to study perturbations of the models, as we do in Section 3.2.

gives  $u(x) = \sqrt{x}$ . Inserting a complete set of eigenstates for the momentum operator  $p$ , Eq. (8) becomes

$$w(l, \mu) = \int_{-\infty}^{+\infty} dp \int_{\mu}^{\infty} dx e^{-lp^2 - \sqrt{x}l} = \frac{2\sqrt{\pi}}{l^{5/2}} e^{-\sqrt{\mu}l} [1 + \sqrt{\mu}l], \quad (9)$$

and the mean area is

$$\bar{A} = -\frac{d}{d\mu} \log w(l, \mu) = \frac{1}{2} \frac{l^2}{1 + \sqrt{\mu}l}. \quad (10)$$

To take the thermodynamic limit we must tune  $l$  and  $\mu$  in such a way that the average area  $\bar{A}$  diverges. Introducing a dimensionless scaling variable  $z = \sqrt{\mu}l$ , we can do this by taking  $l$  to infinity in *three* different ways:  $z \rightarrow 0$ ,  $z = \text{const}$ , and  $z \rightarrow \infty$ , respectively. For the first two cases Eq. (10) gives

$$\bar{A} \sim l^2. \quad (11)$$

In these two cases the boundary length diverges sufficiently slowly that, in the thermodynamic limit, a vanishing fraction of the disk is on the boundary. Combining this relation with Eq. (7) gives Hausdorff dimension  $d_H = 4$ . This agrees with other analytical calculations [5, 10, 20], and numerical simulations [1, 2], for pure gravity.

The third limit,  $z \rightarrow \infty$ , gives  $\bar{A} \sim l$  implying  $d_H = 2$ . This, on the other hand, corresponds in the thermodynamic limit to a disk with a finite fraction of the area close to the boundary, akin to a Bethe lattice. This is the branched polymer phase of the model.

### 3.2 The Ising model ( $m = 3$ )

The Ising model, or the  $(4, 3)$ -model, was solved as a two-matrix model in [21], and the disk amplitude with fixed boundary condition was calculated in [9, 22, 8]. Goulian [9], in fact, calculated the disk amplitude for the Ising model off criticality. Since the thermal perturbation plays a crucial role in our subsequent analysis, we will review this calculation here.

In the planar limit, with vanishing external magnetic field, the differential operators  $P$  and  $Q$ , for the Ising model in the high-temperature phase, are

$$\begin{aligned} Q &= (d^2 - u)_+^{3/2} = d^3 - \frac{3}{2}ud, \\ P &= Q_+^{4/3} - tQ_+^{2/3} = d^4 - 2ud^2 + \frac{1}{2}u^2 - t(d^2 - u), \end{aligned} \quad (12)$$

where  $t$  is the deviation from the critical temperature. The corresponding string equation is

$$x = \frac{1}{2}u^3 + \frac{3}{4}tu^2. \quad (13)$$

The disk amplitude Eq. (8) becomes

$$w(l, \mu) = \int_{\mu}^{\infty} dx \int_{-\infty}^{+\infty} dp e^{-il \left( p^3 + \frac{3}{2} u(x, t) p \right)} \quad (14)$$

$$= \sqrt{\frac{2}{3}} \int_{\mu}^{\infty} dx \sqrt{u(x, t)} K_{1/3} \left( \frac{1}{\sqrt{2}} u^{3/2}(x, t) l \right), \quad (15)$$

where  $K_{\nu}(y)$  is a modified Bessel function of the second kind [23, Eq. 8.433]. Using the string equation Eq. (13) and [23, Eq. 5.52] we get

$$w(l, \mu) = \frac{2}{\sqrt{3}} \frac{u(\mu, t)}{l} \left[ u(\mu, t) K_{4/3}(z) + t K_{2/3}(z) \right], \quad (16)$$

where we have introduced the scaling variable  $z = u^{3/2}(\mu, t) l / \sqrt{2}$ . The mean area is then

$$\bar{A} = \frac{l^2 K_{1/3}(z)}{2 z K_{4/3}(z) + 2^{2/3} t l^{2/3} z^{1/3} K_{2/3}(z)}. \quad (17)$$

Again we consider the physical limits  $z \rightarrow 0$  and  $z = \text{const.}$  We must now specify how the critical point  $t = 0$  should be approached in the continuum limit. In order to obtain scaling characteristic of the bulk, one should approach the critical point with the matter correlation length  $\xi_M$  strictly less than the system size.<sup>3</sup> For the Ising model coupled to gravity  $\xi_M \sim t^{-\nu} = t^{-3/d_H}$ . On the other hand, the linear size of the surface is  $L \equiv A^{1/d_H} \leq l^{2/d_H}$ , where we have used the behavior of the scaling variable  $z$ . Now requiring  $\xi_M \leq L$  implies  $t \geq l^{-2/3}$ . In this limit the second term in the denominator of Eq. (17) dominates, leading to the area scaling

$$\bar{A} \sim l^{4/3}. \quad (18)$$

This together with Eq. (7) yields Hausdorff dimension

$$d_H = 4, \quad (19)$$

which agrees with numerical simulations of the Ising model on dynamical triangulations of spherical topology [1, 2, 3].

If we tune the deviation from the critical temperature more rapidly to zero than  $1/l^{2/3}$ , Eq. (17) gives us the “classical” area scaling  $\bar{A} \sim l^2$ . This implies  $d_H = 6$ , which coincides with the result obtained in [6, 7].

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<sup>3</sup>This approach to the thermodynamic limit is conventional in the theory of finite size scaling [24]. There the critical temperature has to be approached so that  $\xi/L$  is finite, where  $\xi$  is the correlation length and  $L$  the linear system size.

## 4 The continuum limit and Hausdorff dimension

In the previous section we derived the result

$$\bar{A} = \frac{l^2 K_{1/3}(z)}{2 z K_{4/3}(z) + 2^{2/3} t l^{2/3} z^{1/3} K_{2/3}(z)}, \quad (20)$$

where the scaling variable  $z = u^{3/2}(\mu, t)l/\sqrt{2}$ . We now extend our discussion of the nature of the continuum limit in these models. Ultimately we must take the three different continuum limits:

1.  $\mu \rightarrow \mu_c \quad (\bar{A} \rightarrow \infty)$
2.  $\mu_b \rightarrow \mu_b^c \quad (l \rightarrow \infty)$
3.  $t \rightarrow 0 \quad (T \rightarrow T_c)$ .

The first limit corresponds to simply letting the volume of the system diverge. The second limit corresponds to tuning the boundary cosmological constant ( $\mu_b$ ) to its critical value so that the length of the boundary diverges along with the area. And the last limit corresponds to adjusting the temperature so that the Ising spins are critical. If the mean area ( $\bar{A}$ ) diverges like  $\bar{A} \simeq l^\nu$ , as we take  $t \rightarrow 0$ , then we have  $d_H = 3\nu$ , since the geodesic distance scales as  $r \simeq l^{1/3}$  for the Ising model.

The correct procedure, we claim, is to take the limits (1) and (2) first. The situation is analogous to that of the Ising model in a small residual external magnetic field. In that case one is interested in the scaling behavior of the free energy as the scaling variable  $x = h/t^\Delta$  approaches zero. At any small but finite  $h$ , one cannot take the limit  $t \rightarrow 0$ , as one then gets cross-over to the critical behavior associated with the sink  $h = \infty$  (i.e the critical point associated with large  $x$  behavior). Similarly here we cannot take the limit  $t \rightarrow 0$  at any finite  $l$ . In the limit  $l \rightarrow \infty$  first, we have  $\nu = 4/3$  and  $d_H = 4$ . As mentioned, this approach to the continuum limit is also familiar in the theory of finite-size scaling. One has to take the continuum limit in such a way that the matter correlation length is less than the system size, otherwise the physics is dominated by boundary rather than bulk effects.

The line with  $y^{-1} = tl^{2/3}$  held constant may be viewed as analogous to a cross-over line. For  $y$  small, with  $t$  sufficiently small of course to be near the critical point, we have the true critical behavior in which we are interested. For  $y$  large the system is dominated by boundary effects. There should thus be a cross-over for  $y$  in the neighborhood of 1. It is difficult to say exactly how rapid this cross-over is. But eventually one might see the emergence of Hausdorff dimension 6 scaling for  $t$  sufficiently small at finite size  $l$ . It may also be that this regime is never reached. There is no evidence for such a regime in the present numerical simulations but it should be looked for more carefully.

## 5 Discussion

In this paper we examine how the intrinsic dimensionality of two-dimensional gravity depends on its coupling to unitary  $(m+1, m)$  conformal field theories for the case of pure gravity ( $m = 2$ ) and the Ising model ( $m = 3$ ). In such theories one must take the thermodynamic limit simultaneously with the approach to the critical point of the matter. In particular we are interested in the scaling of the mean area of a disk with its boundary length. We have shown that this scaling depends on the precise approach to the infinite volume critical point. In the continuum limit one has to consider how the correlation length of the matter fields diverges in relation to the gravitational correlation length. We find that for the Ising model perturbed by a thermal operator the mean area does not scale according to the classical result  $A \sim l^2$ , but rather with an anomalous exponent that depends on the matter. Combined with the scaling of geodesic distance with boundary length, following from  $c < 1$  string field theory arguments, we find that the Hausdorff dimension of surfaces in  $2d$ -gravity coupled to the Ising is 4. Tuning to the critical temperature before taking the infinite volume limit yields  $d_H = 6$ .

Our results suggest some further avenues of investigation. A critical identification in this paper is that between geodesic distance and proper time in the string field theory – this may simply be wrong when matter is coupled. For a recent discussion of this point and a determination of the Hausdorff dimension for  $c = -2$  matter see [25]. The behavior of minimal models (coupled to gravity) on the disk is intriguing from several points of view, particularly if one adds a boundary magnetic field [26]. Our predicted scaling behavior of the mean area with boundary length on the disk should certainly be checked in numerical simulations.

Although we have restricted our analysis to the case of the pure gravity and the Ising model coupled to gravity these issues remain relevant to the generic  $(p, q)$  minimal model coupled to gravity. There, however, the space of relevant operators is much larger and more intricate and this will lead to a subtle interplay of the different length scales introduced by various perturbations. As we have shown, though, the situation is complicated enough for the Ising model, and a better understanding of even this simple case would greatly clarify the manner in which geometrical quantities behave in reparametrization invariant theories like quantum gravity.

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